MATH 2020 A
Fall 2020

Lecture 17

In the last Lecture we showed the equivalence between conservative v.f. and v.f. indept of path. This is nice. But it doesn't solve one of our main question, namely, how do we know a v.f. is conservative? the condition

$$\oint_C \vec{F}\cdot d\vec{r} = 0 \qquad \forall \text{ closed loop } C$$

involves the testing for ∞-many C and is impossible to carry out!

Here we shall follow a rather common mathematical reasoning to find out some necessary conditions.

Consider $n=2$ first. Here $\vec{F}=(P,Q)$ where P,Q are C^1 in the interior of some region. In case it admits a potential Φ, we've

$$P=\frac{\partial \Phi}{\partial x}, \quad Q=\frac{\partial \Phi}{\partial y},$$

Hence

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

the compatibility condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

does not involve the unknown Φ! It must hold if $\vec{F}$ is conservative.

Theorem 1 Let $\vec{F}$ be C^1-conservative inside a region, then

$n=2$,
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

holds. For $n=3$,

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},$$

holds.

Pf: $n=3$, $\vec{F}=(P,Q,R)$. $\vec{F}=\nabla\Phi$. then

$$\frac{\partial P}{\partial y}=\frac{\partial^2\Phi}{\partial y\partial x}=\frac{\partial^2\Phi}{\partial x\partial y}=\frac{\partial Q}{\partial x},$$

$$\frac{\partial P}{\partial z}=\frac{\partial^2\Phi}{\partial z\partial x}=\frac{\partial^2\Phi}{\partial x\partial z}=\frac{\partial R}{\partial x},$$

$$\frac{\partial Q}{\partial z}=\frac{\partial^2\Phi}{\partial z\partial y}=\frac{\partial^2\Phi}{\partial y\partial z}=\frac{\partial R}{\partial y}.$$

e.g 1. Determine if the v.f. $(2x-3, z, \cos z)$ is conservative or not.

Here $P=2x-3$, $Q=z$, $R=\cos z$. We've

$$\frac{\partial P}{\partial y}=0=\frac{\partial Q}{\partial x},$$

$$\frac{\partial P}{\partial z}=0=\frac{\partial R}{\partial x}, \quad \text{but}$$

$$\frac{\partial Q}{\partial z}=1\neq 0=\frac{\partial R}{\partial y},$$

so this v.f. is not conservative.

e.g. 2 How about the v.f.

$$\frac{-y}{x^2+y^2}\hat{i}+\frac{x}{x^2+y^2}\hat{j} \quad ?$$

Here $P = -y/x^2+y^2$, $Q = x/x^2+y^2$.

$$\frac{\partial P}{\partial y} = \frac{-1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Since $\partial P/\partial y = \partial Q/\partial x$, we can't decide whether it is conservative or not.

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At this point we take a digression to look at the setting: We've been working on v.f in the interior of a region. Hence the geometry of the boundary of the region, or its smoothness, are of no relevance in our discussion. It is useful to formulate our result in a more general setting.

A subset G in $\mathbb{R}^2, \mathbb{R}^3$ is open if $\forall p \in G$, $\exists$ a ball $B_\delta(p) \subset G$, where

$$B_\delta(p) = \{q : |q-p| < \delta\}$$ the ball center at p with radius δ.

Roughly, the regions we have been considering are not open sets, but, after removing their boundaries, they form open set. For instance, the ball

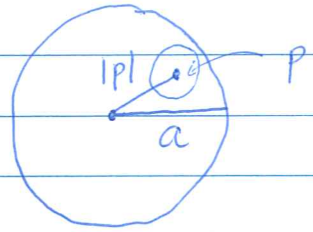
$$\{(x,y,z) : x^2+y^2+z^2 \le a^2\} \quad \text{or} \quad \text{the disk}$$

$$\{(x,y) : x^2+y^2 \le a^2\}$$

is not open, but their interior

$$\{(x,y,z) : x^2+y^2+z^2<a^2\} \quad \text{and}$$

$$\{(x,y) : x^2+y^2<a^2\}$$



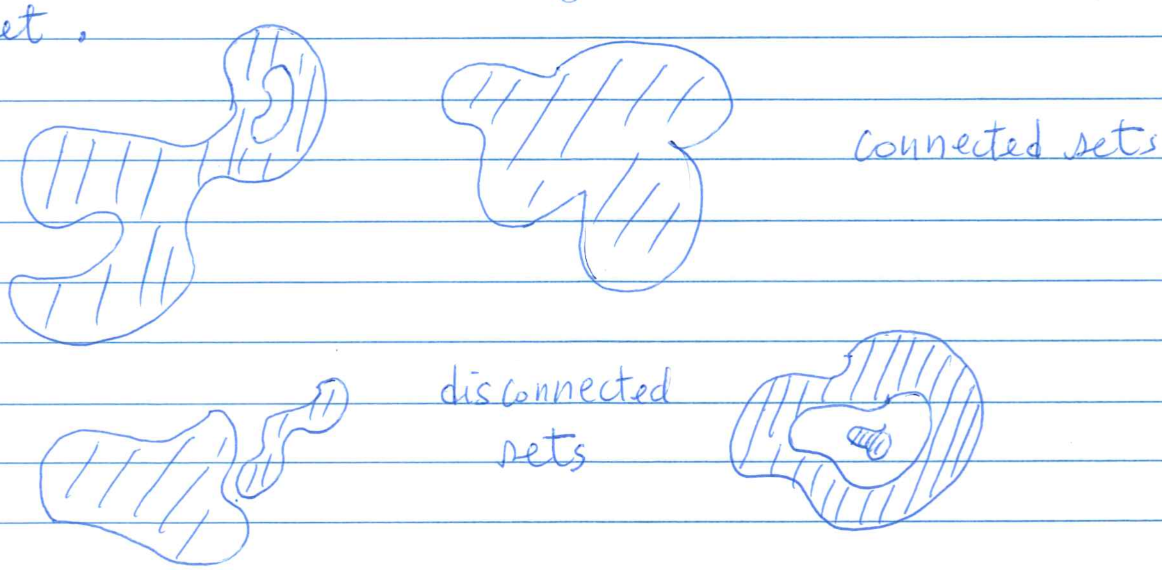
are open sets. For example, let

$p \in$ the interior of the disk, that is, $p_1^2+p_2^2<a^2$. Taking

$$\delta = a-|p|>0,$$

the disk $B_\delta(p) \subset \{(x,y) : x^2+y^2<a^2\}$. Hence $\{(x,y) : x^2+y^2<a^2\}$ is open.

On the other hand, a set is connected if any two pts in this set can be connected by a regular curve lying inside this set.



In our proofs of the theorems concerning conservative v.f's we only use ① v.f. defined in the interior of a region, and ② any 2 pts inside the region can be connected by a piecewise regular curve.

Hence, we have the following re-formulation of the theorems:

Theorem 2 Let $\vec{F}$ be a C^1-v.f. in an open, connected set in $\mathbb{R}^2, \mathbb{R}^3$. Then the followings are equivalent:

(a) $\vec{F}$ is conservative,

(b) $\vec{F}$ is indept of path,

(c) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed loop C.

Theorem 1 also holds for C^1-v.f. in an open, connected set.

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Let's return to e.g. 2. The v.f.

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

satisfies the compatibility condition $\partial P/\partial y = \partial Q/\partial x$. Is there a potential for it?

Let $\theta = \tan^{-1} \frac{y}{x} \in (-\pi, \pi]$. $\therefore$ θ is fcn of (x, y).

From $\tan\theta = y/x$, we get

$$\sec^2\theta \frac{\partial \theta}{\partial x} = -y/x^2, \quad \sec^2\theta \frac{\partial \theta}{\partial y} = 1/x.$$

Using $1+\tan^2\theta = \sec^2\theta = 1 + (y/x)^2 = (x^2+y^2)/x^2$, we have

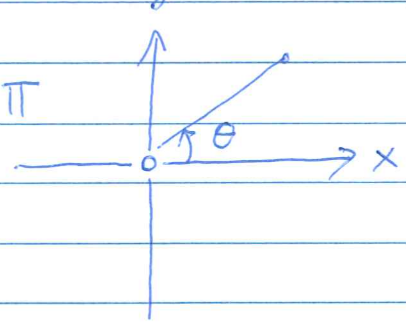
$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}.$$

One may conclude that $\vec{F}$ is conservative because $\theta = \theta(x,y)$ is its potential. However, things are not so simple.

First of all, the natural domain of definition of $\vec{F}$ is

$$\mathbb{R}^2 \setminus \{(0,0)\} = \{(x,y) : 0 < x^2+y^2\}$$

since $|\vec{F}(x,y)| \to \infty$ as $(x,y) \to (0,0)$. However, the function $\theta(x,y)$ is not well-defined in $\mathbb{R}^2 \setminus \{(0,0)\}$!



When θ increases from 0 to π, $\theta(x,y) \uparrow \pi$

so for pts $(x,0)$ on the negative x-axis,

$$\theta(x,0) = \pi, \quad x<0.$$

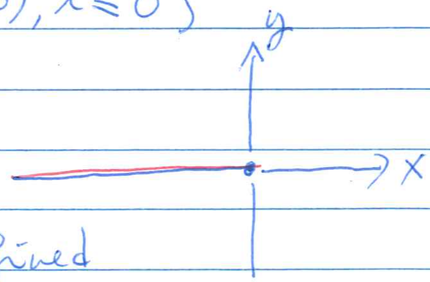
However, when θ decreases from 0 to $-\pi$, $\theta(x,y) \downarrow -\pi$, so

$$\theta(x,0) = -\pi, \quad x<0.$$

θ has a jump discontinuity across the negative x-axis, that's, the "potential fcn" $\theta(x,y)$ is not continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$. But we require a potential function to be C^2 in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Now, if we consider a smaller open, connected set, say

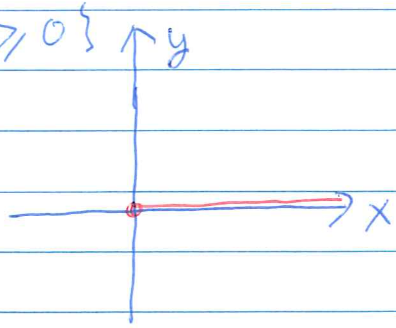
$$D = \{(x,y) : (x,y) \neq (x,0), x \leq 0\}$$



that's $\mathbb{R}^2$ removing the non-positive x-axis. Then $\theta = \theta(x,y)$ is well-defined and ~~θ~~ is the potential for the v.f. $\vec{F}$ in D.

Similarly, if we consider

$$G = \{(x,y) : (x,y) \neq (x,0),\ x \geq 0\}$$



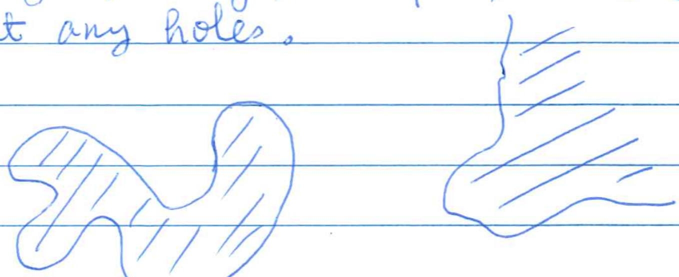
$\mathbb{R}^2$ minus the non-negative x-axis,

a potential function is given by

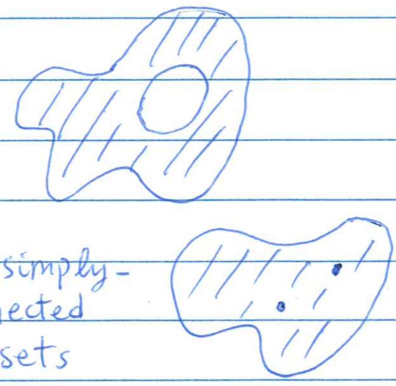
$$\theta = \tan^{-1} y/x \in [0, 2\pi) .$$

This example shows us a very interesting phenomenon, namely, the existence of a potential depends not only on the v.f. itself, but also on the domain it is defined.

A connected set in $\mathbb{R}^2, \mathbb{R}^3$ is simply-connected or contractible if every closed loop inside this set can be continuously deformed to a point where the whole process happens inside this set. Roughly speaking, an open, simply-connected set is a set without any holes.



simply-connected sets



nonsimply-connected sets

3-dim examples will be given in next chapter.

<u>Theorem 3</u> <u>Let</u> $\vec{F}$ <u>be a</u> C^1-v.f. <u>in an open, simply-connected set in</u> $\mathbb{R}^2, \mathbb{R}^3$. <u>Then</u> it is <u>conservative</u> <u>if and only if</u>

$(n=2)$. $\quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\quad$ <u>or</u>

$(n=3)$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},$$

holds.

Theorem 3 follows from Green's theorem $(n=2)$ and Stokes' Thm $(n=3)$. Will explain later.

Let's return to the v.f.

$$\vec{F}=\left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$$

it is defined in $\mathbb{R}^2\setminus\{(0,0)\}$ which is not simply-connected (it is punctured). Theorem 3 asserts that it does not admit a potential. However, both

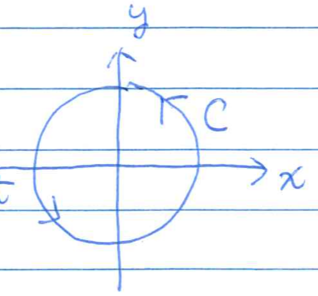
$\mathbb{R}^2 \setminus$ nonpositive x-axis and

$\mathbb{R}^2 \setminus$ nonnegative x-axis

are simply-connected, so $\vec{F}$ is conservative in these 2 sets.

You may ask, well, θ is not a potential for $\vec{F}$ in $\mathbb{R}^2\setminus\{(0,0)\}$ maybe there is another. We may carry out another test to show that it has no potential by using Condition (c) in Theorem 2.

Let $C: \gamma(t)=(\cos t, \sin t)$, $t\in[-\pi,\pi]$



$$\oint_C \vec{F}\cdot d\vec{r}=\int_{-\pi}^{\pi}\left(\frac{-r\sin t}{r^2},\ \frac{r\cos t}{r}\right)\cdot(-\sin t, \cos t)\,dt$$

$$=\int_{-\pi}^{\pi}(\sin^2 t+\cos^2 t)\,dt = 2\pi \neq 0.$$

For several thousand years, geometry and algebra are two different subjects. Geometry is concerned with figures and algebra with symbolic operations. However, with the introduction of Cartesian coordinates, geometry is linked to algebra :

a circle $\leadsto$ $x^2+y^2=a^2$,

a straight line $\leadsto$ $y=ax+b$, etc.

A vector field consists of differentiable functions and so belongs to the realm of analysis. However, whether it is conservative depends on the shape of the underlying space, More precisely, does the space have a hole or not. Again, we see the linkage ~~..~~ between analysis and geometry (more precise topology). The interplay between analysis (v.f, solution of differential equations, etc) and topology (homotopy, cohomology, etc) is a main theme in modern mathematics.

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To conclude the chapter, we point out that the expressions

$$P\,dx + Q\,dy, \qquad P\,dx+Q\,dy+R\,dz,$$

are called differential forms. When there is a potential f, we write

$$df = P\,dx + Q\,dy \qquad n=2$$

$$df = P\,dx+Q\,dy+R\,dz \qquad n=3$$

these differential forms are called exact.

e.g. Show $y\,dx+x\,dy+4\,dz$ is exact and find its potential.

$y\hat{i}+x\hat{j}+4\hat{k}$ is defined - $\mathbb{R}^3$. $\frac{\partial P}{\partial y}=1=\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z}=0=\frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z}=0=\frac{\partial R}{\partial y}$

so it is exact. $f(x,y,z)=xy+4z$.

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx+x\,dy+4\,dz = f(2,3,-1)-f(1,1,1) = 2-5=-3. \#$$

The GREEN'S THEOREM

The fundamental theorem of Calculus is encoded in a single formula

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

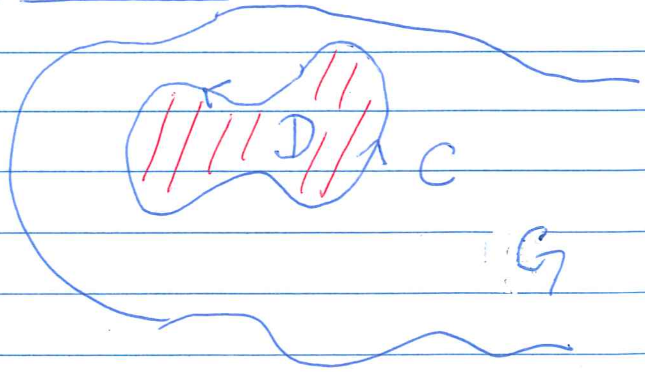
Here the integration over an interval (1-d) is expressed as the evaluation at 2 endpoints (0-d). Green's theorem can be viewed as the generalization of the fundamental theorem. Now the double integral (2-d) is linked to a line integral (1-d).

There are different formulation of the Green's theorem, which apply to various situation. We start with the simplest one.

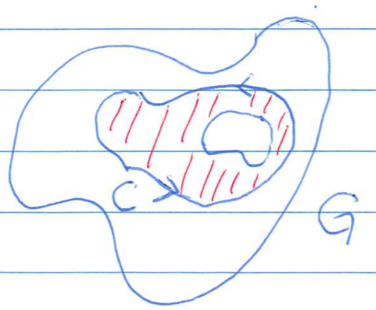
Theorem 1 (Green's thm) Let $\vec{F}$ be a C^1-v.f. in some open set $G \subseteq \mathbb{R}^2$. Let C be a simple, piecewise regular curve in G such that its enclosed set D belongs to G. Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P\,dx + Q\,dy, \quad \text{where}$$

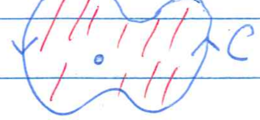
$\vec{F} = (P, Q)$ and C is oriented in anticlockwise direction.



gd.

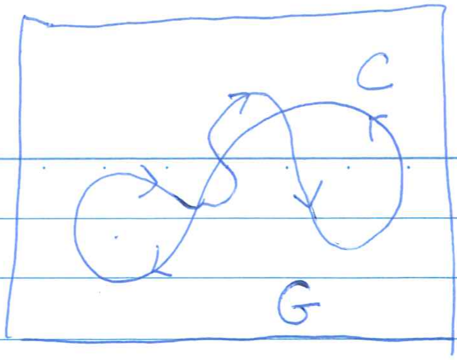


no gd. G has a hole and C encloses the hole, so D not belong to G.



$G = \mathbb{R}^2 \setminus (0,0)$

C encloses (0,0), so D not belong to G. No gd.



no gd. C has self-intersection. No clear how to define D.

We'll not give a complete proof of this theorem, but will do it in some important special case.

First, we consider type A region, if it is given by

$$D = \{ (x,y) : f_1(x) < y < f_2(x),\ x \in (a,b) \}$$

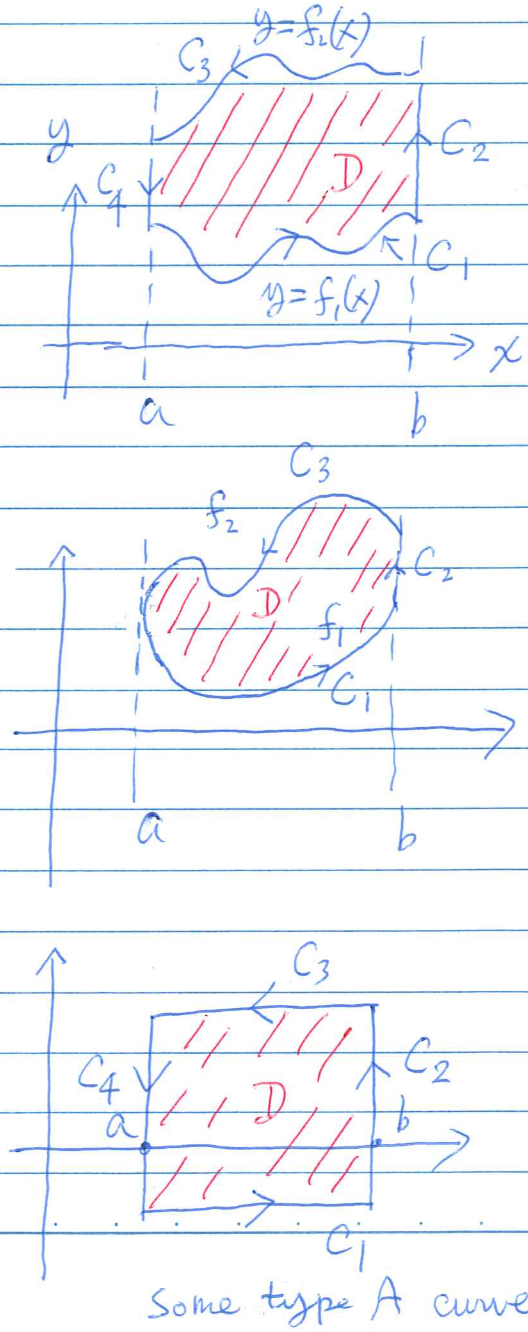
C_1 $\gamma_1(x) = (x, f_1(x)),\ x \in [a,b]$

C_2 $\gamma_2(y) = (b, y),\ y \in [f_1(b), f_2(b)]$

C_3 $\gamma_3(x) = (x, f_2(x)),\ x \in [a,b]$

C_4 $\gamma_4(y) = (a, y),\ y \in [f_1(a), f_2(a)]$

$$C = C_1 + C_2 - C_3 - C_4.$$



Some type A curves.

Claim:

$$\iint_D \frac{\partial P}{\partial y}\, dA = -\oint_C P\, dx \quad (1)$$

LHS $$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b P_1(x,y) \Big|_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_a^b P_1(x, f_2(x)) - P_1(x, f_1(x)) \, dx$$

$$= \int_a^b P_1(x, f_2(x))\, dx - \int_a^b P_1(x, f_1(x))\, dx .$$

$$-\text{RHS} : \oint_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{-C_3} P\,dx + \int_{-C_4} P\,dx$$

$$= \int_a^b P(x, f_1(x)) \cdot 1\, dx + \int_{f_1(b)}^{f_2(b)} P(b,y) \cdot 0 \, dy$$

$$- \int_a^b P(x, f_2(x)) \cdot 1 \, dx - \int_{f_1(a)}^{f_2(a)} P(a,y) \cdot 0 \, dy$$

$$= \int_a^b P(x, f_1(x))\, dx - \int_a^b P(x, f_2(x))\, dx$$

$\therefore$

LHS $=$ $-$RHS.

A type B region is

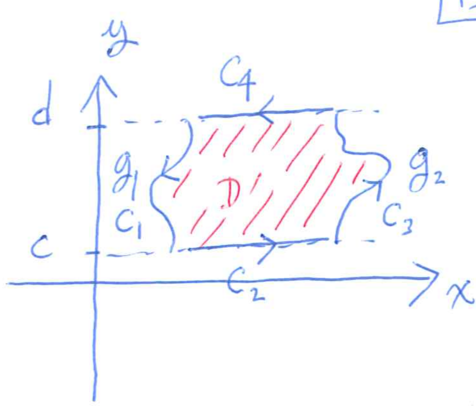
$$D' = \{ (x,y) : g_1(y) < x < g_2(y) ,\ y \in (c,d) \}$$

C_1 $\gamma_1(y) = (g_1(y), y)$, $y \in [c, d]$,

C_2 $\gamma_2(x) = (x, c)$, $x \in [g_1(c), g_2(c)]$,

C_3 $\gamma_3(y) = (g_2(y), y)$, $y \in [c, d]$

C_4 $\gamma_4(x) = (x, d)$, $x \in [g_1(d), g_2(d)]$



$C = -C_1 + C_2 + C_3 - C_4$.

claim: $\displaystyle\iint_{D'} \frac{\partial Q}{\partial x}\, dA = \oint_C Q\, dy$ (2)

LHS $\displaystyle\iint_{D'} \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x}(x,y)\, dx\, dy$

$$= \int_c^d Q(x,y) \Big|_{x=g_1(y)}^{x=g_2(y)} dx$$

$$= \int_c^d Q(g_2(y), y)\, dy - \int_c^d Q(g_1(y), y)\, dy.$$

RHS $\displaystyle\oint_C Q\, dy = -\int_{C_1} Q\, dy + \int_{C_2} Q\, dy + \int_{C_3} Q\, dy - \int_{C_4} Q\, dy$

$$= -\int_c^d Q(g_1(y), y) \cdot 1\, dy + \int_{g_1(c)}^{g_2(c)} Q(x, c) \cdot 0\, dx$$

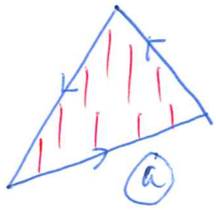
$$+ \int_c^d Q(g_2(y), y) \cdot 1\, dy - \int_{g_1(d)}^{g_2(d)} Q(x, d) \cdot 0\, dx$$

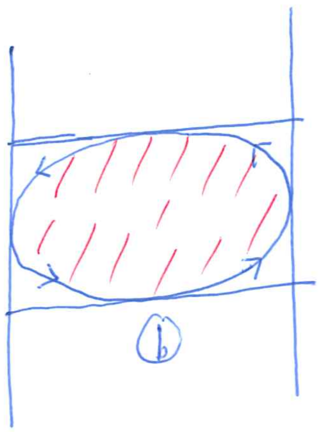
$$= -\int_c^d Q(g_1(y), y)\, dy + \int_c^d Q(g_2(y), y)\, dy .$$

$\therefore$ LHS = RHS.

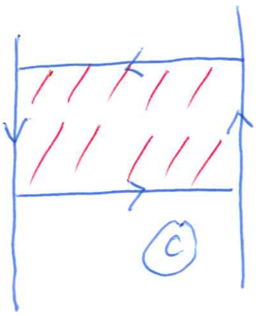
When a region is of type B and A simultaneously, we can add ① and ② together to get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy .$$





a, b, c are of type A and type B.

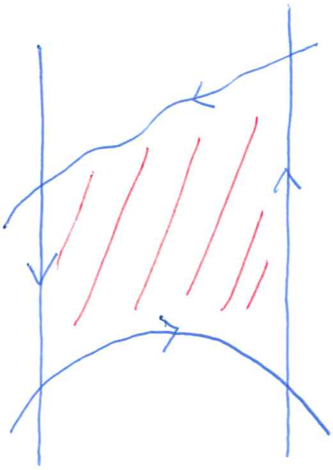


You could find a complete proof of Green's thm in the internet. For a general shape, I would reason in the following way:

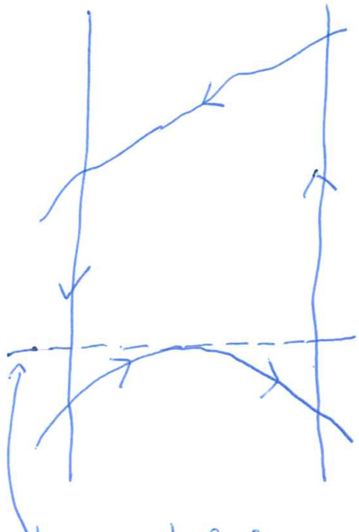
(1) Show Green's thm holds when the region is of type A (not nec. of type B).

(2) Decompose a general shape into finitely many regions of type A.

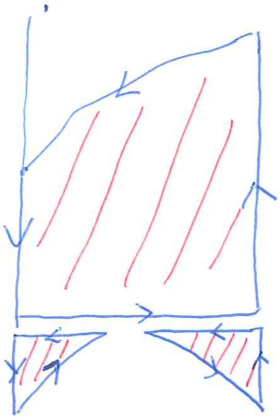
Step 1



type A not
type B

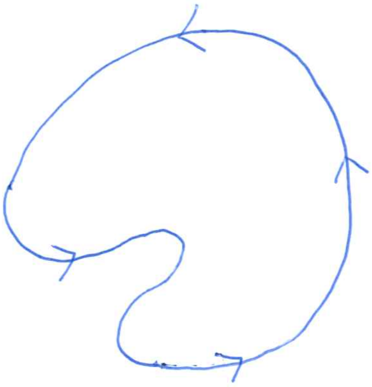


horizontal line
moving down

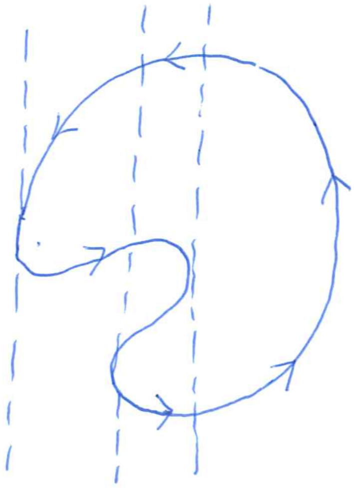


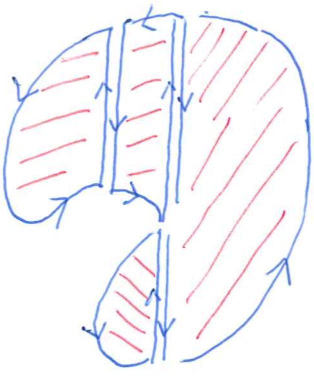
decompose to
3 type A & B regions

Step 2



not type A
not type B





decompose to
4 type A regions.